ON THE THEORY OF AN ABSOLUTELY ELASTIC IMPACT ON MATERIAL SYSTEMS

(K TEORII ABSOLIUTNO UPRUGOGO UDARA MATERIAL'NYKH SISTEM)

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An investigation of an absolutely elastic impact arising in a system on which a unilateral constraint is imposed, initiated in [1], is here continued. A certain property of the impact connected with a minimum of a certain function is derived, and a generalization of the theory to embrace the case of arbitrary smooth constraints is also presented.

1. We shall consider a system of n material points, subjected to smooth time-independent constraints whose equations are

$$f_\alpha(x_1, \ldots, x_{3n}) = 0 \tag{1.1}$$

where $x_1, \ldots, x_{3n}$ are the coordinates of the points with respect to a certain fixed Cartesian coordinate system (x_1, x_2, x_3 are the coordinates of the first point, x_4, x_5, x_6 are the coordinates of the second point, and so on).

At a certain instant of time we impose on the moving system a smooth unilateral constraint

$$\varphi(x_1, \ldots, x_{3n}) \geqslant 0 \tag{1.2}$$

The system then experiences an impact. While investigating the impact in [1] the following equation was used by the author:

$$\sum m_i (v_i - v_{i0})\, \delta x_i \geqslant 0$$

where m is the mass of a point ($m_1 = m_2 = m_3$ is the mass of the first point, $m_4 = m_5 = m_6$ is the mass of the second point, and so on); v_{i0} and v_i are the velocities of a point immediately before and immediately after the impact, respectively; δx_i are the "possible" displacements of the system at the instant of impact. The quantities δx_i satisfy the relations

$$\sum \frac{\partial f_\alpha}{\partial x_i} \delta x_i = 0$$

and also the auxiliary relations which arise from the constraints (1.2)

$$\sum \frac{\partial \varphi}{\partial x_i} \delta x_i \geqslant 0$$

In order to complete the system of conditions of the impact the principle of conservation of the kinetic energy [vis viva] has been used. It has been demonstrated that under these conditions the jumplike changes of the velocity of each point caused by the impact are expressed by

$$v_i - v_{i0} = \mu R_i \tag{1.3}$$

where

$$\mu = 2 \sum \frac{\partial \varphi}{\partial x_i} v_{i0} / \Sigma\, m_i R_i^2 \tag{1.4}$$

$$R_i = \frac{1}{m_i}\left(\sum \frac{A_{\beta\alpha}}{A} \frac{\partial f_\alpha}{\partial x_i} a_\beta - \frac{\partial \varphi}{\partial x_i}\right) \tag{1.5}$$

besides $A_{\alpha\beta}$ are the algebraic cofactors of the elements $a_{\alpha\beta}$ in the determinant $A = |a_{\alpha\beta}|$, and $a_{\alpha\beta}$ and a_β are given by

$$a_{\alpha\beta} = \sum \frac{1}{m_i} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_i}, \qquad a_\beta = \sum \frac{1}{m_i} \frac{\partial f_\beta}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \tag{1.6}$$

It is easy to show that the quantities R_i satisfy the following identities:

$$\sum \frac{\partial f_\alpha}{\partial x_i} R_i = 0, \qquad \sum \frac{\partial \varphi}{\partial x_i} R_i = -\sum m_i R_i^2 \tag{1.7}$$

Indeed

$$\sum \frac{\partial f_\alpha}{\partial x_i} R_i = \sum \frac{\partial f_\alpha}{\partial x_i} \frac{1}{m_i}\left(\sum \frac{A_{\rho\sigma}}{A} \frac{\partial f_\sigma}{\partial x_i} a_\rho - \frac{\partial \varphi}{\partial x_i}\right)$$
$$= \sum \frac{A_{\rho\sigma}}{A} a_{\alpha\sigma} a_\rho - a_\alpha = a_\alpha - a_\alpha = 0$$

and further

$$\sum m_i R_i^2 = \sum R_i \left(\sum \frac{A_{\beta\alpha}}{A} \frac{\partial f_\alpha}{\partial x_i} a_\beta - \frac{\partial \varphi}{\partial x_i}\right)$$
$$= \sum \frac{A_{\beta\alpha}}{A} a_\beta \frac{\partial f_\alpha}{\partial x_i} R_i - \sum \frac{\partial \varphi}{\partial x_i} R_i = -\sum \frac{\partial \varphi}{\partial x_i} R_i$$

on the strength of the above identities. This is what was required. The conditions used in [1] for the problem of an absolutely elastic impact were completed with the condition of conservation of the kinetic energy. However, this is not the only condition which can be used to obtain the desired results. We shall prove now the following theorem:

Theorem. After an impact the real state of a system satisfies the relation

$$\sum \frac{\partial \varphi}{\partial x_i} v_i = -\sum \frac{\partial \varphi}{\partial x_i} v_{i0} \tag{1.8}$$

and differs from other possible states consistent with the constraints (1.8) in that it also satisfies the equation

$$\sum m_i (v_i - v_{i0}) \delta x_i = 0 \tag{1.9}$$

with all δx_i subjected to the conditions

$$\sum \frac{\partial f_\alpha}{\partial x_i} \delta x_i = 0, \qquad \sum \frac{\partial \varphi}{\partial x_i} \delta x_i = 0 \tag{1.10}$$

Indeed, by (1.3) we have

$$\sum \frac{\partial \varphi}{\partial x_i} v_i = \mu \sum \frac{\partial \varphi}{\partial x_i} R_i + \sum \frac{\partial \varphi}{\partial x_i} v_{i0}$$

If we substitute in the above equation the values of μ as given by (1.4), and if we take into account the last identity from (1.7), we verify the validity of Equation (1.8).

Thus the first part of the theorem is proved. To prove the second part it is sufficient to establish that the state of the system after the impact is uniquely determined by the conditions of the theorem and that this state is real.

Introducing undetermined multipliers λ_α and μ we derive from (1.9) and from (1.10)

$$m_i (v_i - v_{i0}) + \sum \lambda_\alpha \frac{\partial f_\alpha}{\partial x_i} + \mu \frac{\partial \varphi}{\partial x_i} = 0 \tag{1.11}$$

Substituting the values of v_i as found in (1.11) in the equations resulting from (1.1)

$$\sum \frac{\partial f_\alpha}{\partial x_i} v_i = 0$$

and using symbols from (1.6) we obtain

$$\sum \lambda_\alpha a_{\alpha\beta} + \mu a_\beta = 0$$

Since the determinant $A = |a_{\alpha\beta}|$ does not vanish we have

$$\lambda_\alpha = -\mu \sum \frac{A_{\beta\alpha}}{A} a_\beta \qquad (1.12)$$

Therefore Equations (1.11) can be transformed into

$$v_i - v_{i0} = \mu R_i \qquad (1.13)$$

where R_i are in accordance with (1.5).

In order to find μ we use (1.13) to eliminate the quantities v_i from Equation (1.8). We obtain

$$\mu \sum \frac{\partial \varphi}{\partial x_i} R_i = -2 \sum \frac{\partial \varphi}{\partial x_i} v_{i0}$$

If we take into account the last identity from (1.7) we can see that the expression for μ resulting from it agrees with (1.4). Thus the theorem is proved.

The above theorem leads in turn to some interesting conclusions. If the velocities v_i and v_i^* vary and assume all values, real or otherwise, which the constraints (1.8) permit after the impact, then on the strength of (1.10) we have

$$\delta x_i = v_i^* - v_i$$

Hence Equation (1.9) can be written in the form

$$\sum m_i (v_i - v_{i0})(v_i^* - v_i) = 0$$

Thus, using the identities

$$-(v_i - v_{i0})(v_i^* - v_i) = \tfrac{1}{2}[(v_i - v_{i0})^2 - (v_i^* - v_{i0})^2 + (v_i - v_i^*)^2]$$

we find

$$\sum \frac{m_i}{2}(v_i - v_{i0})^2 - \sum \frac{m_i}{2}(v_i^* - v_{i0})^2 + \sum \frac{m_i}{2}(v_i - v_i^*)^2 = 0$$

and obtain finally

$$\sum \frac{m_i}{2}(v_i - v_{i0})^2 \leqslant \sum \frac{m_i}{2}(v_i^* - v_{i0})^2$$

In this way, among all the states consistent with the constraints and which satisfy the condition (1.8), the real state is the one for which the function

$$\sum \frac{m_i}{2}(v_i - v_{i0})^2$$

assumes the minimum.

2. Let us assume for the moment that the constraint imposed on the system remains after the impact.

The impact resulting from imposing a bilateral constraint on the system has been investigated in detail in [2,3]. It is governed by the equation

$$\sum m_i (u_i - v_{i0})\, \delta x_i = 0$$

where u_i are the velocities of the system after the impact and δx_i are subjected to the conditions

$$\sum \frac{\partial f_\alpha}{\partial x_i} \delta x_i = 0, \qquad \sum \frac{\partial \varphi}{\partial x_i} \delta x_i = 0$$

Following the steps taken in the previous paragraph almost exactly, we find that the velocities after the impact are expressed by the equations

$$u_i - v_i = \nu R_i \tag{2.1}$$

which are analogous to the equations (1.3), with the difference that the undetermined multiplier in this case is

$$\nu = -\sum \frac{\partial \varphi}{\partial x_i} v_{i0} \Big/ \sum \frac{\partial \varphi}{\partial x_i} R_i \tag{2.2}$$

The above formula has been derived from the equation

$$\sum \frac{\partial \varphi}{\partial x_i} u_i = 0$$

which in turn has been obtained from the bilateral constraint (1.2) by eliminating u_i through (2.1).

Now let the bilateral constraint again become unilateral. Let the system which responded to this constraint be subjected to impulses $m_i \nu R_i$ which equal the reactions caused by the constraints (the constraint (1.2) among others), as was so in the other case. We want to find the state reached by the system caused by this second impact.

Here we deal with a system subjected to an impulsive action with some of the constraints unilateral. This kind of impact has been investigated by Mayer [4]. Following Mayer we shall seek the minimum of the expression

$$\sum \frac{m_i}{2} (v_i - u_i - \nu R_i)^2 \tag{2.3}$$

with conditions

$$\sum \frac{\partial f_\alpha}{\partial x_i} v_i = 0, \qquad \sum \frac{\partial \varphi}{\partial x_i} v_i \geqslant 0 \tag{2.4}$$

The presence of an inequality among the conditions (2.4) means that the region D of the possible values v_i determined by this inequality is bounded. This fact introduces a certain special feature, namely, that the problem can be solved by initially neglecting the inequality and is then checked if v_i obtained in this way satisfies the neglected inequality (2.4) or not. If it does, then the investigation is finished; if it does not, then the minimum of (2.3) must be sought on the boundary of the region D. From (2.3) and (2.4) we find

$$m_i (v_i - u_i - \nu R_i) + \sum \lambda_\alpha \frac{\partial f_\alpha}{\partial x_i} = 0$$

In order to find the undetermined multipliers λ_α from these last equations we eliminate v_i from Equations (2.4); we obtain

$$\sum \lambda_\alpha a_{\alpha\beta} - \sum \frac{\partial f_\beta}{\partial x_i} u_i - \sum \frac{\partial f_\beta}{\partial x_i} \nu R_i = 0$$

Since $a_{\alpha\beta}$ are as given by (1.6) the above system has a non-vanishing determinant, and because of the identities

$$\sum \frac{\partial f_\beta}{\partial x_i} u_i = 0, \qquad \sum \frac{\partial f_\beta}{\partial x_i} R_i = 0$$

the free terms equal zero. It means that all $\lambda_\alpha = 0$. It means also that

$$v_i - u_i = \nu R_i. \tag{2.5}$$

Substituting the resulting values of v_i in the left member of the inequality (2.4) we obtain

$$\sum \frac{\partial \varphi}{\partial x_i} v_i = \sum \frac{\partial \varphi}{\partial x_i} u_i + \nu \sum \frac{\partial \varphi}{\partial x_i} R_i = \nu \sum \frac{\partial \varphi}{\partial x_i} R_i = -\sum \frac{\partial \varphi}{\partial x_i} v_{i0} \geqslant 0$$

since the inequality

$$\sum \frac{\partial \varphi}{\partial x_i} v_{i0} \leqslant 0$$

must be satisfied before the impact.

In this way, the state of the system after the impact, as expressed by (2.5), satisfies the inequality (2.4); hence, as we have just observed, the state of the system must be real. The resultant of the two considered impacts is found by eliminating u_i from Equations (2.1) and (2.5). It is expressed by the equation

$$v_i - v_{i0} = 2\nu R_i$$

which happens to be the same as the one from an impact caused by a unilateral constraint. (This can be seen by comparing Formula (2.2) with (1.4)). This fact leads to an interesting conclusion. Let us re-examine briefly our procedure. We investigated two impacts one after another. The first one was caused by imposing on the system a new constraint which was assumed not to vanish after the impact. After the first impact occurred the constraint was again made unilateral, and the system was then subjected to impulses equalling the reactions caused by the first impact. It was found that after the second impact the system assumed the same state it would have reached if the constraint (1.2) had been unilateral from the beginning.

Unifying both impacts into a single process and giving the impacts a physical interpretation, we can say that an impact represented by a unilateral constraint consists of a sequence of two phases. The first is the non-elastic one, when the impact is propagated absolutely non-elastically and the reactions accumulate, the second is an elastic one where the impulse of the reactions accumulated in the first phase explosively releases the system of the unilateral constraints.

3. The physical interpretation of the absolutely elastic impact has been formulated on the assumption that the constraints are time-independent, and that a constraint imposed on a system is expressed by a single inequality. The obviousness of this physical interpretation suggests that in the general case of holonomic or nonholonomic constraints expressed through more than one inequality the resulting impact would also behave the same way.

This is, of course, a hypothesis. When accepted, it permits the calculation in a general case of the state of a system after an impact. So doing we consider the impact in two consecutive phases and proceed as shown above, or use the method given in [5].

The results obtained in the first section of this paper are applicable also to a general case.

The state of a system after the impact satisfies the relations

$$\sum b_{\mu i} v_i + b_\mu = -(\sum b_{\mu i} v_{i0} + b_\mu) \tag{3.1}$$

if the unilateral constraints imposed on the system are expressed by the inequalities

$$\sum b_{\mu i} v_i + b_\mu \geqslant 0$$

The actual state differs from all other states of the system permitted by the constraints and satisfying the relations (3.1) by the fact that all δx_i subjected to the conditions

$$\sum c_{\lambda i}\, \delta x_i = 0, \qquad \sum b_{\mu i}\, \delta x_i = 0 \qquad (3.2)$$

must satisfy the equations

$$\sum m_i (v_i - v_{i0})\, \delta x_i = 0$$

The first relation in (3.2) comes from the equations of constraints and the second one results from Equations (3.1), which express the elastic properties of the constraints.

From the theorem previously proved it follows that after the impact, among all possible states of a system permitted by the constraints and satisfying (3.1), the state which is real is the one for which the function

$$\sum \frac{m_i}{2} (v_i - v_{i0})^2$$

assumes a minimum.

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